

## COMMON FIXED POINT THEOREMS IN HILBERT SPACE

ARVIND KUMAR SHARMA<sup>1</sup>, V. H. BADSHAH<sup>2</sup> & V. K. GUPTA<sup>3</sup>

<sup>1</sup>Department of Engineering Mathematics, M.I.T.M, Ujjain, Madhya Pradesh, India

<sup>2</sup>School of Studies in Mathematics, Vikram University, Ujjain, Madhya Pradesh, India

<sup>3</sup>Department of Mathematics, Government Madhav Science College, Ujjain, Madhya Pradesh, India

### ABSTRACT

The aim of this present paper is to obtain a common fixed point for continuous self mappings on Hilbert space. Our purpose here is to generalize the our previous result [3]

**KEYWORDS:** Common Fixed Point, Hilbert Space, Continuous Self Mappings

**2010 Mathematics Subject Classification:** 47H10.

### INTRODUCTION

Kannan [3] proved that a self mapping  $T$  on complete metric space  $(X, d)$  satisfying the condition

$$d(x, y) \leq \alpha \{ d(x, Tx) + d(y, Ty) \} \text{ for all } x, y \text{ in } X \text{ where } 0 < \alpha < \frac{1}{2} \text{ has a unique fixed in } (X, d).$$

Koparde and Waghmode [5] have proved fixed point theorem for a self mappings  $T$  on a closed subset  $S$  of Hilbert space  $H$ , satisfying the Kannan type condition

$$\|Tx - Ty\|^2 \leq \alpha \left\{ \|x - Tx\|^2 + \|y - Ty\|^2 \right\} \text{ for all } x, y \text{ in } S \text{ and } x \neq y; 0 \leq \alpha < \frac{1}{2}$$

Koparde and Waghmode [5] also extended this result to the pair of mappings  $T_1$  and  $T_2$  and to their power  $p, q$  are some positive integers.

In this paper we have obtained a unique fixed point for self mapping satisfying rational inequality

$$\|Tx - Ty\| \leq \alpha \left\{ \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{\|x - Tx\| + \|y - Ty\|} \right\} + \beta \|x - y\| \text{ for all } x, y \text{ in } S \text{ and } x \neq y; \frac{1}{2} > \alpha \geq 0, \beta \geq 0 \text{ and } 2\alpha + \beta < 1.$$

In a Hilbert space

### MAIN RESULTS

We proved followingg fixed point theorems.

**Theorem 1:** Let  $S$  be a closed subset of a Hilbert space  $H$ . Let  $T$  be a self mappings on  $S$  satisfying the following condition

$$\|Tx - Ty\| \leq \alpha \left\{ \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{\|x - Tx\| + \|y - Ty\|} \right\} + \beta \|x - y\| \text{ for all } x, y \text{ in } S \text{ and } x \neq y; \frac{1}{2} > \alpha \geq 0, \beta \geq 0 \text{ and } 2\alpha + \beta < 1.$$

Then  $T$  has a unique common fixed point.

**Proof:** Let  $S$  be a closed subset of a Hilbert space  $H$ . Let  $T$  be a self mappings on  $S$ . Let  $x_0 \in S$  be any arbitrary point in  $S$ .

Define a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $S$  by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \text{ for } n = 0, 1, 2, \dots$$

Suppose that  $x_{n+1} \neq x_n$  for  $n = 0, 1, 2, \dots$

For any integer  $n \geq 1$ .

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|Tx_n - Tx_{n-1}\| \\ &\leq \alpha \left\{ \frac{\|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_{n-1}\|^2}{\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\|} \right\} + \beta \|x_n - x_{n-1}\| \\ &= \alpha \frac{\|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_n\|^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} + \beta \|x_n - x_{n-1}\| \\ &\leq \alpha \frac{\{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|\}^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} + \beta \|x_n - x_{n-1}\| \\ &\leq \alpha \|x_n - x_{n+1}\| + \alpha \|x_{n-1} - x_n\| + \beta \|x_n - x_{n-1}\| \\ &= \alpha \|x_{n+1} - x_n\| + \alpha \|x_n - x_{n-1}\| + \beta \|x_n - x_{n-1}\| \\ \text{i.e. } \|x_{n+1} - x_n\| &\leq \alpha \|x_{n+1} - x_n\| + \alpha \|x_n - x_{n-1}\| + \beta \|x_n - x_{n-1}\| \\ \Rightarrow (1 - \alpha) \|x_{n+1} - x_n\| &\leq (\alpha + \beta) \|x_n - x_{n-1}\| \\ \Rightarrow \|x_{n+1} - x_n\| &\leq \frac{\alpha + \beta}{1 - \alpha} \|x_n - x_{n-1}\| \end{aligned}$$

If  $k = \frac{\alpha + \beta}{1 - \alpha}$  then  $k < 1$ .

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq k \|x_n - x_{n-1}\| \\ &\leq k \|x_n - x_{n-1}\| \leq k^2 \|x_{n-1} - x_{n-2}\| \leq k^3 \|x_{n-2} - x_{n-3}\| \leq \dots \leq k^n \|x_1 - x_0\| \end{aligned}$$

i.e.  $\|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\|$  for all  $n \geq 1$  is integer.

Now for any positive integer  $m \geq n \geq 1$

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \dots + \|x_{m-1} - x_m\| \\ &\leq k^n \|x_1 - x_0\| + k^{n+1} \|x_1 - x_0\| + \dots + k^{m-1} \|x_1 - x_0\| \end{aligned}$$

$$\leq k^n \|x_1 - x_0\| \left(1 + k + \dots + k^{m-n-1}\right)$$

$$i.e. \|x_n - x_m\| \leq \left(\frac{k^n}{1-k}\right) \|x_1 - x_0\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (} k < 1 \text{)}$$

Therefore  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence.

Since S is a closed subset of a Hilbert space H, so  $\{x_n\}_{n=1}^{\infty}$  converges to a point u in S.

Now we will show that u is common fixed point of self mappings T from S into S.

Suppose that  $Tu \neq u$ .

Consider

$$\|u - Tu\| \leq \|u - x_n\| + \|x_n - Tu\|$$

$$= \|x_n - Tu\|$$

$$= \|Tx_{n-1} - Tu\|$$

$$\leq \alpha \left\{ \frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|u - Tu\|^2}{\|x_{n-1} - Tx_{n-1}\| + \|u - Tu\|} \right\} + \beta \|x_{n-1} - u\|$$

$$\leq \alpha \left\{ \frac{\|x_{n-1} - x_n\|^2 + \|u - Tu\|^2}{\|x_{n-1} - x_n\| + \|u - Tu\|} \right\} + \beta \|x_{n-1} - u\|$$

$$i.e. \quad \|u - Tu\| \leq \alpha \left\{ \|x_{n-1} - x_n\| + \|u - Tu\| \right\} + \beta \|x_{n-1} - u\|$$

$$(1-\alpha)\|u - Tu\| \leq \alpha \|x_{n-1} - x_n\| + \beta \|x_{n-1} - u\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{So } \|u - Tu\| \leq 0.$$

$$\text{Hence } u = Tu.$$

Hence u is a common fixed point of self mappings T.

**Uniqueness:** Suppose that there is  $u \neq w$  such that  $Tu = u$  and  $Tw = w$ .

$$\text{Consider } \|u - w\| = \|Tu - Tw\|$$

$$\leq \alpha \left\{ \frac{\|u - Tu\|^2 + \|w - Tw\|^2}{\|u - Tu\| + \|w - Tw\|} \right\} + \beta \|u - w\|$$

$$\leq \alpha \|u - Tu\| + \alpha \|w - Tw\| + \beta \|u - w\|$$

$$i.e. \quad \|u - w\| \leq \beta \|u - w\|$$

$$\Rightarrow \|u - w\| = 0$$

Thus  $u = w$ .

Hence fixed point is unique.

**Theorem 2:** Let  $S$  be a closed subset of a Hilbert space  $H$ . Let  $T_1$  and  $T_2$  are self mappings on  $S$  satisfying the following condition

$$\|T_1x - T_2y\| \leq \alpha \left\{ \frac{\|x - T_1x\|^2 + \|y - T_2y\|^2}{\|x - T_1x\| + \|y - T_2y\|} \right\} + \beta \|x - y\| \text{ for all } x, y \text{ in } S \text{ and } x \neq y; \frac{1}{2} > \alpha \geq 0, \beta \geq 0 \text{ and } 2\alpha + \beta < 1.$$

Then  $T_1$  and  $T_2$  have a unique common fixed point.

**Proof:** Let  $S$  be a closed subset of a Hilbert space  $H$ . Let  $T_1$  and  $T_2$  are self mappings on  $S$ . Let  $x_0 \in S$  be any arbitrary point in  $S$ .

Define a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $S$  by

$$\left. \begin{aligned} x_{2n+1} &= T_1x_{2n} \\ x_{2n+2} &= T_2x_{2n+1} \end{aligned} \right\} \text{ for } n = 0, 1, 2, \dots$$

Suppose that  $x_{n+2} \neq x_{n+1} \neq x_n$  for  $n = 0, 1, 2, \dots$

Consider

$$\begin{aligned} \|x_1 - x_2\| &= \|T_1x_0 - T_2x_1\| \\ &\leq \alpha \left\{ \frac{\|x_0 - T_1x_0\|^2 + \|x_1 - T_2x_1\|^2}{\|x_0 - T_1x_0\| + \|x_1 - T_2x_1\|} \right\} + \beta \|x_0 - x_1\| \\ &\leq \alpha \frac{\|x_0 - x_1\|^2 + \|x_1 - x_2\|^2}{\|x_0 - x_1\| + \|x_1 - x_2\|} + \beta \|x_0 - x_1\| \\ &\leq \alpha \|x_0 - x_1\| + \alpha \|x_1 - x_2\| + \beta \|x_0 - x_1\| \\ \text{i.e. } \|x_1 - x_2\| &\leq \alpha \|x_0 - x_1\| + \alpha \|x_1 - x_2\| + \beta \|x_0 - x_1\| \\ \Rightarrow (1 - \alpha) \|x_1 - x_2\| &\leq (\alpha + \beta) \|x_0 - x_1\| \\ \Rightarrow \|x_1 - x_2\| &\leq \frac{(\alpha + \beta)}{(1 - \alpha)} \|x_0 - x_1\| \end{aligned}$$

If  $k = \frac{\alpha + \beta}{1 - \alpha}$  then  $k < 1$ .

$$\|x_1 - x_2\| \leq k \|x_0 - x_1\|$$

Now

$$\begin{aligned}
\|x_2 - x_3\| &= \|T_2x_1 - T_1x_2\| \\
&= \|T_1x_2 - T_2x_1\| \\
&\leq \alpha \left\{ \frac{\|x_2 - T_1x_2\|^2 + \|x_1 - T_2x_1\|^2}{\|x_2 - T_1x_2\| + \|x_1 - T_2x_1\|} \right\} + \beta \|x_2 - x_1\| \\
&\leq \alpha \frac{\|x_2 - x_3\|^2 + \|x_1 - x_2\|^2}{\|x_2 - x_3\| + \|x_1 - x_2\|} + \beta \|x_2 - x_1\| \\
\text{i.e. } \|x_2 - x_3\| &\leq \alpha \|x_2 - x_3\| + \alpha \|x_1 - x_2\| + \beta \|x_2 - x_1\| \\
\Rightarrow (1 - \alpha) \|x_2 - x_3\| &\leq (\alpha + \beta) \|x_1 - x_2\| \\
\Rightarrow \|x_2 - x_3\| &\leq \frac{(\alpha + \beta)}{(1 - \alpha)} \|x_1 - x_2\|
\end{aligned}$$

If  $k = \frac{\alpha + \beta}{1 - \alpha}$  then  $k < 1$ .

$$\begin{aligned}
\|x_2 - x_3\| &\leq k \|x_1 - x_2\| \\
&\leq k^2 \|x_0 - x_1\|
\end{aligned}$$

$$\text{i.e. } \|x_2 - x_3\| \leq k^2 \|x_0 - x_1\|$$

Hence by induction we get for integer  $n \geq 1$

$$\|x_n - x_{n+1}\| \leq k^n \|x_0 - x_1\|$$

or

$$\|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\|$$

Now for any positive integer  $m \geq n \geq 1$

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \dots + \|x_{m-1} - x_m\| \\
&\leq k^n \|x_1 - x_0\| + k^{n+1} \|x_1 - x_0\| + \dots + k^{m-1} \|x_1 - x_0\| \\
&\leq k^n \|x_1 - x_0\| (1 + k + \dots + k^{m-n-1})
\end{aligned}$$

$$\text{i.e. } \|x_n - x_m\| \leq \left( \frac{k^n}{1 - k} \right) \|x_1 - x_0\| \rightarrow 0 \text{ as } n \rightarrow \infty \ (k < 1)$$

Therefore  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence.

Since  $S$  is a closed subset of a Hilbert space  $H$ , so  $\{x_n\}_{n=1}^{\infty}$  converges to a point  $u$  in  $S$ .

Now we will show that  $u$  is common fixed point of the pair of mappings  $T_1$  and  $T_2$  from  $S$  into  $S$ .

First we will show that  $u$  is fixed point of  $T_1$ .

Suppose that  $T_1 u \neq u$

Consider

$$\begin{aligned}
 \|u - T_1 u\| &\leq \|u - x_{2n+2}\| + \|x_{2n+2} - T_1 u\| \\
 &\leq \|x_{2n+2} - T_1 u\| \\
 &= \|T_2 x_{2n+1} - T_1 u\| \\
 &= \|T_1 u - T_2 x_{2n+1}\| \\
 &\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|x_{2n+1} - T_2 x_{2n+1}\|^2}{\|u - T_1 u\| + \|x_{2n+1} - T_2 x_{2n+1}\|} \right\} + \beta \|u - x_{2n+1}\| \\
 &\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|x_{2n+1} - x_{2n+2}\|^2}{\|u - T_1 u\| + \|x_{2n+1} - x_{2n+2}\|} \right\} + \beta \|u - x_{2n+1}\| \\
 \text{i.e.} \quad \|u - T_1 u\| &\leq \alpha \{ \|u - T_1 u\| + \|x_{2n+1} - x_{2n+2}\| \} + \beta \|u - x_{2n+1}\|
 \end{aligned}$$

$$(1 - \alpha) \|u - T_1 u\| \leq \alpha \|x_{2n+1} - x_{2n+2}\| + \beta \|u - x_{2n+1}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{So } \|u - T_1 u\| \leq 0.$$

Hence  $u = T_1 u$ .

Similarly we can prove that  $u$  is fixed point of  $T_2$ .

Hence  $u$  is a common fixed point of mappings  $T_1$  and  $T_2$

**Uniqueness:** Suppose that there is  $w \neq u$  such that  $T_2 w = w$ .

$$\begin{aligned}
 \text{Consider } \|u - w\| &= \|T_1 u - T_2 w\| \\
 &\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|w - T_2 w\|^2}{\|u - T_1 u\| + \|w - T_2 w\|} \right\} + \beta \|u - w\| \\
 &\leq \alpha \|u - T_1 u\| + \alpha \|w - T_2 w\| + \beta \|u - w\| \\
 \text{i.e. } \|u - w\| &\leq \beta \|u - w\| \\
 \Rightarrow \|u - w\| &= 0
 \end{aligned}$$

Thus  $u = w$ .

Similarly  $v \in S$  if is such that  $T_1 v = v$  then  $u = v$

Hence  $u$  is unique common fixed point of  $T_1$  and  $T_2$ .

### Corollary

Let  $p$  and  $q$  be two positive integers such that operators  $T_1^p$  and  $T_2^q$ :  $S \rightarrow S$  where  $S$  is closed subset of Hilbert space  $H$ , satisfy the inequality

$$\|T_1^p x - T_2^q y\| \leq \alpha \left\{ \frac{\|x - T_1^p x\|^2 + \|y - T_2^q y\|^2}{\|x - T_1^p x\| + \|y - T_2^q y\|} \right\} + \beta \|x - y\| \text{ for all } x, y \text{ in } S \text{ and } x \neq y; \frac{1}{2} > \alpha \geq 0, \beta \geq 0 \text{ and } 2\alpha + \beta < 1.$$

Then  $T_1$  and  $T_2$  have a unique common fixed point.

**Proof:** By the above theorem, the  $T_1^p$  and  $T_2^q$  have a unique common fixed point, say  $u \in S$ .

$$\text{Since } u = T_1^p u$$

$$\begin{aligned} \text{therefore } T_1 u &= T_1 (T_1^p u) \\ &= T_1^p (T_1 u) \end{aligned}$$

i.e.  $T_1 u$  is also a fixed point of  $T_1^p$ . But by above theorem  $T_1^p$  has a unique fixed point.

Therefore,  $T_1 u = u$ .

Similarly  $T_2 u = u$ .

Hence  $u$  is common fixed point of  $T_1$  and  $T_2$ .

For uniqueness, if  $w$  is a another common fixed point of  $T_1$  and  $T_2$ , then clearly  $w$  is also a common fixed point of  $T_1^p$  and  $T_2^q$ .

This gives  $w = u$ .

This completes the proof.

**Example:** Let  $X = [0, 1]$ , with Euclidean metric  $d$ . Then  $\{X, d\}$  is a Hilbert space with the norm defined by

$$d(x, y) = \|x - y\|.$$

$$\text{Let } \{x_n\}_{n=1}^{\infty} = \begin{cases} \frac{1}{2 \cdot 6^{(n-1)/2}} x_0 & \text{when } n \text{ is odd} \\ \frac{1}{6^{n/2}} x_0 & \text{when } n \text{ is even} \end{cases}$$

be the sequence in  $X$  and let  $T_1$  and  $T_2$  are self mappings on  $X$  defined by

$$T_1(x) = \frac{x}{3} \quad \text{and} \quad T_2(x) = \frac{x}{2}$$

Clearly  $T_1$  and  $T_2$  are satisfy the rational inequality. Also sequence  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $X$ , which is converges in  $X$  also it has a common point in  $X$ .

## CONCLUSIONS

In this paper we replace Kannan inequality by rational inequality. The theorems proved in this paper by using rational inequality is improved and stronger form of some earlier inequality given by Kannan [3], Koparde and Waghmode [5]. We have obtained a unique fixed point for self mapping satisfying rational inequality extended this result to the pair of mappings  $T_1$  and  $T_2$  and to their power  $p, q$  are some positive integers.

## REFERENCES

1. Badshah, V.H. and Meena, G., (2005): Common fixed point theorems of an infinite sequence of mappings, Chh. J. Sci. Tech. Vol. 2, 87-90.
2. Browder, Felix E. (1965): Fixed point theorem for non compact mappings in Hilbert space, Proc Natl Acad Sci U S A. 1965 June; 53(6): 1272–1276.
3. Kannan, R., (1968) Some results on fixed points, Bull. Calcutta Math. Soc. 60,71-76.
4. Koparde, P.V. and Waghmode, B.B. (1991): On sequence of mappings in Hilbert space, The Mathematics Education, XXV, 197.
5. Koparde, P.V. and Waghmode, B.B. (1991): Kannan type mappings in Hilbert spaces, Scientist Phyl. Sciences Vol.3, No.1, 45-50.
6. Pandhare, D.M. and Waghmode, B.B. (1998): On sequence of mappings in Hilbert space, The Mathematics Education, XXXII, 61.
7. Sangar, V.M. and Waghmode, B.B. (1991): Fixed point theorem for commuting mappings in Hilbert space-I, Scientist Phyl. Sciences Vol.3, No.1, 64-66.
8. Sharma, A.K., Badshah, V.H and Gupta, V.K. (2012): Common fixed point theorems of a sequence of mappings in Hilbert space, Ultra Scientist Phyl. Sciences.
9. Veerapandhi, T. and Kumar, Anil S. (1999): Common fixed point theorems of a sequence of mappings in Hilbert space, Bull. Cal. Math. Soc.91 (4), 299-308.